MATHEMATICS

A NOTE ON SEPARATION AXIOMS FOR COMPACT SYNTOPOGENOUS SPACES

BY

D. C. J. BURGESS AND M. FITZPATRICK

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INTRODUCTION

Separation axioms for syntopogenous spaces have been studied by A. Császár [1] and by J. L. Sieber and W. J. Pervin [2, 3]. In [2, 3] it was observed that the T_0, T_1, T_2 and Urysohn separation axioms for a syntopogenous space $[E, \mathscr{S}]$ are purely topological in that they depend on the topology $\mathscr{S}^{tp}$ generated by $\mathscr{S}$ rather than on $\mathscr{S}$ itself. Here, we show that the same can be said for the regular, completely regular, Stone and normal axioms, in the case where $[E, \mathscr{S}]$ is compact.

1. DEFINITIONS

Let $[E, \mathscr{S}]$ denote an arbitrary syntopogenous space and let $\tau(\mathscr{S})$ denote the classical topology associated with $\mathscr{S}^{tp}$. For a subset A of E, the closure of A with respect to $\tau(\mathscr{S})$ will be written as $c(A)$.

Following the terminology of [2, 3] the space $[E, \mathscr{S}]$ is called:

$\mathscr{S}$-completely regular iff for $x \notin c(A)$, $\exists$ an $(\mathscr{S}, \mathscr{H})$-continuous function f of E into R (the reals with the usual uniformity $\mathscr{H}$) such that $f(x)=0$ and $f(y)=1$ for $y \in A$;

$\mathscr{S}$-Stone iff for $x \neq y$, $\exists$ an $(\mathscr{S}, \mathscr{H})$-continuous function f of E into R such that $f(x)=0$ and $f(y)=1$.

For the $\mathscr{S}$-regular and $\mathscr{S}$-normal axioms we adopt the formulations set out below (and clearly equivalent to their counterparts in [2]).

The space $[E, \mathscr{S}]$ is termed:

$\mathscr{S}$-regular iff for $x \notin A$, where A is $\tau(\mathscr{S})$-closed, x and A have disjoint $\mathscr{S}$-neighbourhoods;

$\mathscr{S}$-normal iff for $A \cap B = \emptyset$, where A and B are $\tau(\mathscr{S})$-closed, A and B have disjoint $\mathscr{S}$-neighbourhoods.

2. RESULTS

Lemma. If $[A, \mathscr{S}|A]$ is a compact subspace of $[E, \mathscr{S}]$, then the $\mathscr{S}$-neighbourhoods and the $\mathscr{S}^{tp}$-neighbourhoods of A coincide.

PROOF. If $\mathscr{S}^{tp} = \{<_0\}$, and $A <_0 G$, then, to each $a \in A$, there corresponds (by [1; (4.7), (8.38)]) some $<_a \in \mathscr{S}$ such that $a <_a G$; choose $<'_a \in \mathscr{S}$ and $G_a \subseteq E$ such that $a <'_a G_a <'_a G$. Then, by [1; (15.74), (15.77)], $\exists$ a finite subset $\{a_1, \ldots, a_n\}$ of A such that $A \subseteq \bigcup_1^n G_{a_i}$. If $< (\in \mathscr{S})$ is chosen finer than each $<'_{a_i}$, then $A < G$.

Combining the above lemma and [1; (15.83)] we obtain the following.

COROLLARY. If $[E, \mathscr{S}]$ is compact and A is a $\tau(\mathscr{S})$-closed subset of E, then the $\mathscr{S}$-neighbourhoods and the $\mathscr{S}^{tp}$-neighbourhoods of A coincide.

THEOREM. A compact space $[E, \mathscr{S}]$ is $\mathscr{S}$-regular, $\mathscr{S}$-completely regular, $\mathscr{S}$-Stone or $\mathscr{S}$-normal iff $[E, \mathscr{S}^{tp}]$ is $\mathscr{S}^{tp}$-regular, $\mathscr{S}^{tp}$-completely regular, $\mathscr{S}^{tp}$-Stone or $\mathscr{S}^{tp}$-normal, respectively.

PROOF. Owing to our previous results only the $\mathscr{S}$-completely regular and $\mathscr{S}$-Stone axioms require comment.

Since $[E, \mathscr{S}]$ is compact, then so too is $[E, \mathscr{S}^t]$ [1; 15.78)]. Using the symmetry of $\mathscr{H}^t$, we have (for a given $f: E \to R$) that its $(\mathscr{S}^{tp}, \mathscr{H})$-continuity $\Rightarrow$ its $(\mathscr{S}^{tp}, \mathscr{H}^{tp})$-continuity [1; (10.12)] $\Rightarrow$ its $(\mathscr{S}^t, \mathscr{H}^t)$-continuity [1; (15.89)] $\Rightarrow$ its $(\mathscr{S}^t, \mathscr{H})$-continuity [1; (10.10)].

Our result is then immediate from [2; Theorem 2] and [3; Lemma 1].

The remark in [2] that every compact T_2-space is normal then follows immediately from [1; (15.78)] and the above theorem.

Queen's University of Belfast
Northern Ireland Polytechnic

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